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# An Operator on Families of Univalent Functions

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For $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disk and either univalent or in a proper subclass of the family of univalent functions we investigate various relationships between $f(z)$ and $g(z)=z(f(z)/z)^{\gamma}$, $\gamma$ real. Sufficient conditions on the coefficients of $f$ are given for $g$ to be starlike or convex, and coefficient bounds on $g$ are determined. © 1987 Academic Press, Inc.

## 1. INTRODUCTION

A function

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \tag{1}$$

is said to be in the family $S$ it it is analytic and univalent in the unit disk $\Delta=\{|z|<1\}$. The subfamily of functions starlike of order $\alpha$, denoted by $S^*(\alpha)$, consists of functions $f$ for which $\operatorname{Re}(zf'/f)\geqslant\alpha$, $0\leqslant\alpha\leqslant 1$, for $z\in\Delta$. We further denote by $S_1^*(\alpha)$ the subfamily of $S^*(\alpha)$ consisting of functions $f$ for which $|(zf'/f)-1|\leqslant 1-\alpha$, $0\leqslant\alpha\leqslant 1$, for $z\in\Delta$. It is known [6] that a sufficient condition for $f$ to be in $S_1^*(\alpha)$ is that

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leqslant 1-\alpha \tag{2}$$

and that this condition is also necessary if $f$ is of the form

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n,\qquad a_n\geqslant 0. \tag{3}$$

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The subfamily of $S^*(\alpha)$ consisting of functions of the form (3) is denoted by $T^*(\alpha)$ $(\subset S_1^*(\alpha))$. Finally, a function $f$ of the form (1) is said to be in $K$, the family of convex functions, if $\operatorname{Re}\{1+zf''/f'\} \geqslant 0$ for $z \in \Delta$.

In this note we investigate the relationship between $f$ and

$$g(z) = g_\gamma(z) = z(f(z)/z)^\gamma, \qquad \gamma \text{ real}, \tag{4}$$

when $f$ is in $S$ or in one of the above subclasses. In Section 2 we find sufficient coefficient conditions on $f$ for $g$ to be in $S^*(\alpha)$, $S_1^*(\alpha)$, or $K$. The sufficient conditions found by Reade, Silverman, and Todorov [3] for functions of the form $z/(1+\sum_{n=1}^{\infty} b_n z^n)$ to be in $S^*(\alpha)$ or $K$ follow from ours upon setting $\gamma = -1$ in (4). In Section 3 we find necessary and sufficient conditions for $g$ to be in $S^*(\alpha)$, $S_1^*(\alpha)$, or $T^*(\alpha)$ in terms of the corresponding $f$. This leads to coefficient bounds on $g$. In Section 4, we show that $f \in S$ implies $g \in S$ only for $\gamma = 0, 1$.

In the sequel we shall assume, unless otherwise stated, that $f$ is of the form (1) with corresponding $g$ of the form (4), $\gamma$ real. Note that the trivial case $\gamma = 0$, where $g(z) \equiv z$ for any $f$, may be omitted from consideration.

## 2. Sufficient Conditions

THEOREM 1. *The function $g$ is in $S^*(\alpha)$ if*

$$\sum_{n=2}^{\infty} (|\gamma|\,(n-1) + |\gamma(n-1) + 2(1-\alpha|)\,|a_n| \leqslant 2(1-\alpha). \tag{5}$$

*Proof.* Set $p(z) = zg'(z)/g(z) = 1 + \gamma((zf'(z)/f(z)) - 1)$ and $q(z) = (1 - (p(z)-\alpha)/(1-\alpha))/(1 + (p(z)-\alpha)/(1-\alpha))$. Then $\operatorname{Re} p(z) \geqslant \alpha$ for $z \in \Delta$ if and only if $|q(z)| \leqslant 1$. But

$$\begin{aligned} q(z) &= \left| \frac{-\gamma((zf'/f)-1)}{2(1-\alpha) + \gamma((zf'/f)-1} \right| \\ &= \left| \frac{-\gamma \sum_{n=2}^{\infty} (n-1)\, a_n z^{n-1}}{2(1-\alpha) + \sum_{n=2}^{\infty} [\gamma(n-1) + 2(1-\alpha)\, a_n]\, z^{n-1}} \right| \\ &\leqslant \frac{\sum_{n=2}^{\infty} |\gamma|\,(n-1)\,|a_n|}{2(1-\alpha) - \sum_{n=2}^{\infty} |\gamma(n-1) + 2(1-\alpha)\,|a_n|}. \end{aligned}$$

Now inequality (5) is equivalent to this last expression being bounded above by (1), and the proof is complete.

The conclusion of Theorem 1 may be broken into three cases, which we state as

COROLLARY 1. *The function g is in* $S^*(\alpha)$ *if*

(i) $\sum_{n=2}^{\infty} [\gamma(n-1)+(1-\alpha)]\,|a_n| \leqslant 1-\alpha, \qquad \gamma>0,$

(ii) $\sum_{n=2}^{\infty} [|\gamma|\,(n-1)-(1-\alpha)]\,|a_n| \leqslant 1-\alpha,\ \gamma \leqslant -2(1-\alpha),$

(iii) $\sum_{n=n_0+1}^{\infty} [|\gamma|\,(n-1)-(1-\alpha)]\,|a_n| \leqslant (1-\alpha)(1-\sum_{n=2}^{n_0} |a_n|),$
$-2(1-\alpha)<\gamma<0,$

*where* $n_0$ *is the smallest integer for which* $n_0 \geqslant 2(1-\alpha)/|\gamma|$. *Equality holds when* $f(z)=z+(1-\alpha)\,z^n/(|\gamma|\,(n-1)+(1-\alpha))$ *for* $n \geqslant 2$ *in the first two cases and for* $n \geqslant n_0+1$ *in the third.*

The following special case of Corollary 1 may be found in [3].

COROLLARY 2. *The function* $f(z)=z/(1+\sum_{n=2}^{\infty} b_n z^n)$ *is in* $S^*(\alpha)$, $0 \leqslant \alpha \leqslant 1$, *if*

$$\sum_{n=2}^{\infty} (n-1+\alpha)\,|b_n| \leqslant \begin{cases} (1-\alpha)-(1-\alpha)\,|b_1|, & 0\leqslant\alpha\leqslant\frac{1}{2},\\ (1-\alpha)-\alpha\,|b_1|, & \frac{1}{2}\leqslant\alpha\leqslant 1.\end{cases}$$

*Proof.* Set $\gamma=-1$ and $a_n=b_{n-1}$ in Corollary 1.

THEOREM 2. *The function g is in* $S_1^*(\alpha)$ *if*

$$\sum_{n=2}^{\infty} [|\gamma|\,(n-1)+(1-\alpha)]\,|a_n| \leqslant 1-\alpha, \tag{6}$$

*with equality for* $f(z)=z+(1-\alpha)\,z^n/(|\gamma|\,(n-1)+(1-\alpha))$, $n \geqslant 2$.

*Proof.* We have

$$\left|\frac{zg'}{g}-1\right| = \left|\gamma\left(\frac{zf'}{f}-1\right)\right| = \left|\frac{\sum_{n=2}^{\infty}\gamma(n-1)\,a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n z^{n-1}}\right|$$
$$\leqslant \left|\frac{|\gamma|\sum_{n=2}^{\infty}(n-1)\,|a_n|}{1-\sum_{n=2}^{\infty}|a_n|}\right| \leqslant 1-\alpha$$

if and only if (6) holds.

*Remark.* Since $S_1^*(\alpha) \subset S^*(\alpha)$, the special case $\gamma>0$ in Theorem 1 is a consequence of Theorem 2.

THEOREM 3. *The function g is in K if for some a,* $0 \leqslant a \leqslant 1$, *we have*

(i) $\sum_{n=2}^{\infty} [|\gamma-1|\,(n-1)+a]\,|a_n| \leqslant a$ *and*

(ii) $\sum_{n=2}^{\infty} (n-a)\,|\gamma(n-1)+1|\,|a_n| \leqslant 1-a.$

*Proof.* A simple computation yields

$$\begin{aligned} r(z) &= 1 + zg''(z)/g'(z) \\ &= 1 + \frac{(\gamma-1)\sum_{n=2}^{\infty}(n-1)\,a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n z^{n-1}} \\ &\quad + \frac{\sum_{n=2}^{\infty}(n-1)(\gamma(n-1)+1)\,a_n z^{n-1}}{1+\sum_{n=2}^{\infty}(\gamma(n-1)+1)\,a_n z^{n-1}}. \end{aligned}$$

A sufficient condition for $\operatorname{Re} r(z) \geqslant 0$ in $\Delta$ is that the inequality

$$\left|\frac{\sum_{n=2}^{\infty}(\gamma-1)(n-1)\,a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n z^{n-1}}\right| \leqslant \frac{\sum_{n=2}^{\infty}|\gamma-1|\,(n-1)\,|a_n|}{1-\sum_{n=2}^{\infty}|a_n|} \leqslant a$$

and the inequality

$$\begin{aligned} &\left|\frac{\sum_{n=2}^{\infty}(n-1)(\gamma(n-1)+1)\,a_n z^{n-1}}{1+\sum_{n=2}^{\infty}(\gamma(n-1)+1)\,a_n z^{n-1}}\right| \\ &\quad\leqslant \frac{\sum_{n=2}^{\infty}(n-1)\,|\gamma(n-1)+1|\,|a_n|}{1-\sum_{n=2}^{\infty}|\gamma(n-1)+1|\,|a_n|} \leqslant 1-a \end{aligned}$$

both hold for some $a$, $0 \leqslant a \leqslant 1$. But these inequalities are equivalent to (i) and (ii) being satisfied, and the proof is complete.

When $\gamma > 0$ we may choose the real number $a$ in Theorem 3 large enough so that

$$[(n-1)\,|\gamma-1| + a]/a \leqslant (n-a)(\gamma(n-1)+1)/(1-a) \tag{7}$$

holds for $n \geqslant 2$, which means that condition (ii) implies condition (i). We then choose the smallest such value of $a$, leading to the following single inequality.

COROLLARY 1. *The function $g$ is in $K$ if $\sum_{n=2}^{\infty}[\gamma n^2 + (1-\gamma-\gamma a)n - a(1-\gamma)]\,|a_n| \leqslant 1-a$, where $a = (2+\gamma-\sqrt{5\gamma^2+4})/2\gamma$ when $0 < \gamma \leqslant 1$ and $a = 3/2 - \sqrt{5/4 + 1/\gamma}$ when $\gamma > 1$. Equality holds for $f(z) = z + (1-a)\,z^2/(\gamma+1)(2-a)$.*

*Remark.* The special case $\gamma = 1$ in Theorems 1 and 2 reduces to the sufficient condition (2) for $f$ to be in $S^*(\alpha)$ and $S_1^*(\alpha)$, respectively, and to the well-known sufficient condition for convexity, $\sum_{n=2}^{\infty} n^2 |a_n| \leqslant 1$, in Corollary 1 of Theorem 3.

When $\gamma < 0$, inequality (7) need not hold for all $n$. In fact, the right side of (3) vanishes when $\gamma = -1/(n-1)$. For $\gamma < -\frac{1}{2}$ we can still find a reasonably nice expression in which, for some $a$, inequality (7) holds for $n \geqslant 3$ but not for $n = 2$. This leads to

COROLLARY 2. *The function $g$ is in $K$ for $\gamma < -\frac{1}{2}$ if* $((1+|\gamma|+a)/a)|a_2| + \sum_{n=3}^{\infty} (n-a)[|\gamma|(n-1)-1]/(1-a) \leqslant 1$, *where*

$$a = \begin{cases} (\sqrt{12\gamma^2+8\gamma+5}+1+4\gamma)/2(1+\gamma), & \gamma \neq -1, \\ 2/3 & \gamma = -1. \end{cases}$$

The following special case of Corollary 2 may be found in [3].

COROLLARY 3. *The function $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n)$ is in $K$ if*

$$4|b_1| + \sum_{n=2}^{\infty} (n-1)(3n+1)|b_n| \leqslant 1.$$

*Proof.* Set $\gamma = -1$, $a = \frac{2}{3}$, and $a_n = b_{n-1}$ in Corollary 2.

## 3. RELATIONSHIP BETWEEN CLASSES

THEOREM 4. (i) *The function $f$ is in $S^*(\alpha)$ if and only if* $g \in S^*(1-\gamma(1-\alpha))$, $0 \leqslant \gamma(1-\alpha) \leqslant 1$, *and $g \in S^*(\alpha)$ if and only if* $f \in S^*(1-(1-\alpha)/\gamma)$, $(1-\alpha)/\gamma \leqslant 1$; (ii) *the function $f$ is in $S_1^*(\alpha)$ if and only if* $g \in S_1^*(1-|\gamma|(1-\alpha))$, $|\gamma|(1-\alpha) \leqslant 1$, *and $g \in S_1^*(\alpha)$ if and only if* $f \in S_1^*(1-(1-\alpha)/|\gamma|)$, $(1-\alpha)/|\gamma| \leqslant 1$.

*Proof.* The first result follows from the identity $zg'/g = 1 + \gamma((zf'/f)-1)$ and the second from the identity $|(zg'/g)-1) = |\gamma|\,|(zf'/f)-1|$.

COROLLARY 1. *For $0 \leqslant \gamma \leqslant 1$, $g \in S^*(\alpha)$ whenever $f \in S^*(\alpha)$ and for $|\gamma| \leqslant 1$, $g \in S_1^*(\alpha)$ whenever $f \in S_1^*(\alpha)$.*

The coefficient bounds on $f$ in $S^*(\alpha)$ and $S_1^*(\alpha)$ lead to corresponding coefficient bounds on $g$.

COROLLARY 2. *If $f \in S^*(\alpha)$, $0 \leqslant \gamma(1-\alpha) \leqslant 1$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then $|b_n| = \prod_{k=2}^{n} [(k-2)+2\gamma(1-\alpha)]/(n-1)!$. Equality holds for $g_0(z) = z(f_0(z)/z)^\gamma$, where $f_0(z) = z/(1-z)^{2(1-\alpha)}$.*

*Proof.* The function $f_0(z)$ is known to maximize the coefficients of functions in $S^*(\alpha)$, $0 \leqslant \alpha \leqslant 1$. See [4 or 5].

COROLLARY 3. *If $f \in S_1^*(\alpha)$, $|\gamma|(1-\alpha) \leqslant 1$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then $|b_n| \leqslant |\gamma|(1-\alpha)/(n-1)$. Equality holds for $g_n(z) = z(f_n(z)/z)^\gamma$, where $f_n(z) = z\exp((1-\alpha)\, z^{n-1}/(n-1))$.*

*Proof.* In [7] the function $f_n(z)$ was shown to maximize the $n$th coefficient for functions in $S_1^*(\alpha)$, $0 \leqslant \alpha \leqslant 1$.

In the previous two corollaries the extremal $f$ in $S^*(\alpha)$ and $S_1^*(\alpha)$ was transformed into a $g$ that was extremal in $S^*(1-\gamma(1-\alpha))$ and $S_1^*(1-|\gamma|(1-\alpha))$, respectively. This made the determination of coefficient bounds on $g$ quite straightforward. We now consider a special subclass of $S_1^*(\alpha)$ for which this is not the case. Since $T^*(\alpha) \subset S_1^*(\alpha)$, it follows from Theorem 4 that if $f \in T^*(\alpha)$ then $g \in S_1^*(1-|\gamma|(1-\alpha))$, $|\gamma|(1-\alpha) \leqslant 1$. The extremal functions $g_n$ for the coefficients, however, were not associated with corresponding functions $f \in T^*(\alpha)$. To determine such coefficient bounds, we will need the following lemma, which may be found in [2].

LEMMA. *If* $(1+\sum_{n=2}^{\infty} a_n z^n)^{\gamma} = 1 + \sum_{n=2}^{\infty} b_n z^n$ *is analytic in a neighborhood of the origin,* $\gamma$ *real, then* $b_{k+1} = \sum_{j=0}^{k} [\gamma - (\gamma+1) j/(k+1)] a_{k+1-j} b_j$ $(k=0, 1,..., ; b_0 = 1)$.

THEOREM 5. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T^*(\alpha)$, $|\gamma|(1-\alpha) \leqslant 1$, *and* $g(z) = z(f(z)/z)^{\gamma} = z + \sum_{n=2}^{\infty} b_n z^n$, *then* $|b_n| \leqslant |\gamma|(1-\alpha)/(n-\alpha)$. *Equality holds for* $g_n(z) = z(f_n(z)/z)^{\gamma}$, *where* $f_n(z) = z - (1-\alpha) z^n/(n-\alpha)$.

*Proof.* By the lemma, $b_2 = \gamma a_2$ and

$$b_{k+1} = \gamma a_{k+1} + \sum_{j=1}^{k-1} [\gamma - (\gamma-1) j/k] a_{k+1-j} b_{j+1}. \tag{8}$$

In view of (2) we may set $a_k = \lambda_k (1-\alpha)/(k-\alpha)$ with $\sum_{k=2}^{\infty} \lambda_k \leqslant 1$ and write (8) as $b_{k+1} = \gamma \lambda_{k+1}((1-\alpha)/(k+1-\alpha)) + \sum_{j=1}^{k-1} [\gamma - ((\gamma+1) j/k)] \lambda_{k+1-j}((1-\alpha)/(k+1-j-\alpha)) b_{j+1}$. It suffices to show that $|b_{k+1}|$ is uniquely maximized when $\lambda_{k+1} = 1$, which will be true if

$$|\gamma| \left(\frac{1-\alpha}{k+1-\alpha}\right) > \left|\gamma - \frac{(\gamma+1) j}{k}\right| \left(\frac{1-\alpha}{k+1-j-\alpha}\right) |b_{j+1}|, \quad 1 \leqslant j \leqslant k-1. \tag{9}$$

We proceed inductively. Since $|b_2| = \gamma \lambda_2 (1-\alpha)/(2-\alpha) \leqslant |\gamma|(1-\alpha)/(2-\alpha)$, we may assume that

$$|b_j| \leqslant |\gamma|(1-\alpha)/(j-\alpha) \qquad \text{for} \quad j = 1, 2,..., k. \tag{10}$$

Note that

$$\left|\gamma - \frac{(\gamma+1) j}{k}\right| \leqslant \frac{|\gamma|(k-j)+j}{k} \leqslant \frac{(k-j)+j(1-\alpha)}{k(1-\alpha)} \leqslant \frac{k-\alpha}{k(1-\alpha)}. \tag{11}$$

Substituting the upper bounds of (10) and (11) into the right side of (9), it suffices to show that

$$|\gamma| \left(\frac{1-\alpha}{k+1-\alpha}\right) > \left(\frac{k-\alpha}{k(1-\alpha)}\right)\left(\frac{1-\alpha}{k+1-j-\alpha}\right)\left(\frac{|\gamma|(1-\alpha)}{j+1-\alpha}\right). \tag{12}$$

Since the right side of (12) is maximized when $j=1$, inequality (12) will be true if $1/(k+1-\alpha) > 1/k(2-\alpha)$, which is valid for $k>1$. This completes the proof.

*Remark.* If $0 \leqslant \gamma \leqslant 1$, then $g(z) = z(1-\sum_{n=2}^{\infty} a_n z^{n-1})^{\gamma} = z[1 - \gamma(\sum_{n=2}^{\infty} a_n z^{n-1}) - (\gamma(1-\gamma)/2!)(\sum_{n=2}^{\infty} a_n z^{n-1})^2 - \cdots] = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geqslant 0$. Thus, in addition to $g$ being in $S_1^*(1-\gamma(1-\alpha))$, we have $g \in T^*(1-\gamma(1-\alpha))$.

## 4. A Negative Result

While $g_\gamma(z) = z(f(z)/z)^{\gamma}$ usually seems to share many of the nice properties of $f$, at least when $f$ is in different subclasses of $S$, the same does not hold when the only restriction on $f$ is that it be a member of $S$. In this case, $g_\gamma$ need not even be locally univalent.

THEOREM 6. *For every $\gamma$ real, $\gamma \neq 0, 1$, there exists an $f \in S$ for which $g_\gamma(z) = z(f(z)/z)^{\gamma} \notin S$.*

*Proof.* We have

$$g_\gamma'(z) = \left(\frac{f(z)}{z}\right)^{\gamma-1} \left[\frac{(1-\gamma)f(z) + \gamma z f'(z)}{z}\right] = 0$$

if $zf'(z)/f(z) = (\gamma-1)/\gamma$. Since $zf'/f$ maps $\Delta$ onto the right half plane when $f(z) = z/(1-z)^2$, the corresponding $g_\gamma$ will not be univalent when $\gamma < 0$ or $\gamma > 1$. We now consider $\gamma \in (0, 1)$. For every fixed $z \in \Delta$, the region of values of $\log(zf'(z)/f(z))$ for $f \in S$ is the disk $|w| \leqslant \log((1+|z|)/(1-|z|))$. See, for example, [1, p. 331]. In particular, for any real number $t$ we can find $z \in \Delta$ and $f \in S$ for which $\log(zf'(z)/f(z)) = t + \pi i$. Thus, $zf'(z)/f(z) = -e^t = (\gamma-1)/\gamma$ when $t = \log((1-\gamma)/\gamma)$. This completes the proof.

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